The worm process for the Ising model is rapidly mixing

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Collaborators

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- Daniel Tokarev (Monash University)
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- Finite graph $G = (V, E)$
- Choose $\sigma \in \{-1, +1\}^V$ randomly via
  \[ P(\sigma) = \frac{1}{Z} \exp \left( \beta \sum_{uv \in E} \sigma_u \sigma_v + h \sum_{u \in V} \sigma_u \right) \]
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- External field $h$
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- Main physical interest is in certain moments such as:
  - Susceptibility $\chi = \text{var} \left( \sum_{u \in V} \sigma_u \right)$
Computational Complexity

Theorem (Jerrum 1987; Jerrum-Sinclair 1993)

*Computing the Ising partition function is \#P-hard.*
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Despite this intractability, one can still seek for \( \chi \) a **fully-polynomial randomized approximation scheme (fpras):**
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  - with run time polynomial in \( |V|, \epsilon^{-1} \) and \( \eta^{-1} \)
**Markov-chain Monte Carlo**

Method to study intractable distribution $\pi$ on large sample space $\Omega$

- Construct a transition matrix $P$ on $\Omega$ which:
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  - $|\Omega| = 2^{|V|}$ so rapid mixing implies very few states need be visited
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  - No rigorous results previously known
Rapid mixing of worm process

Theorem (Collevecchio, G., Hyndman, Tokarev 2015+)

For any temperature, the relaxation time of the worm process on graph $G = (V, E)$ satisfies

$$t_{\text{rel}} \leq 4\Delta m n^4$$

with $n = |V|$, $m = |E|$ and $\Delta$ the maximum degree.
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Proof is via the path method.
High-temperature expansions and the PS measure

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- If $x = \tanh \beta$ then:
  - Ising susceptibility $\chi = \frac{1}{\pi(C_0)}$
High-temperature expansions and the PS measure

- Worm process simulates a graphical representation of Ising
- Let $C_k = \{A \subseteq E : (V, A) \text{ has } k \text{ odd vertices}\}$
- PS measure defined on the configuration space $C_0 \cup C_2$

$$\pi(A) \propto x^{|A|} \begin{cases} n, & A \in C_0, \\ 2, & A \in C_2. \end{cases}$$

- If $x = \tanh \beta$ then:
  - Ising susceptibility $\chi = \frac{1}{\pi(C_0)}$
- PS measure is stationary distribution of worm process
Worm process

Worm proposals:
Worm process

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- If $A \in C_0$: 

![Diagram of worm process]
Worm process

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Metropolize proposals with respect to PS measure $\pi(\cdot)$
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- We use the **path method**
- Consider the **transition graph** $G = (\mathcal{V}, \mathcal{E})$ of the worm process
  - $\mathcal{V} = \mathcal{C}_0 \cup \mathcal{C}_2$
  - $\mathcal{E} = \{AA' : P(A, A') > 0\}$
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- Specify paths $\gamma_{I,F}$ in $\mathcal{G}$ between pairs of states $I, F$
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Lemma (Schweinsberg (2001))

Consider MC with state space $\Omega$ and stationary distribution $\pi$. Let $S \subseteq \Omega$, and specify paths $\Gamma = \{\gamma_{I,F}: I \in \Omega, F \in S\}$. Then

$$t_{rel} \leq 4 \mathcal{L}(\Gamma) \varphi(\Gamma)$$

where $\mathcal{L}(\Gamma)$ is the length of a longest path in $\Gamma$ and

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- We choose $S = C_0$. Elementary to show $\pi(C_0) \geq 1/n$. 

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- To transition from $I$ to $F$
  - Flip each $e \in I \triangle F$
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Introduction
Main Theorem
Proof
Discussion
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  - $A_i$ disjoint cycles
- $\gamma_{I,F}$ defined by:
  - Traverse $A_0$
  - $\ldots$ then $A_1$
  - $\ldots$ then $A_2$
  - $\ldots$
Choice of Canonical Paths

- To transition from $I$ to $F$
  - Flip each $e \in I \triangle F$
- If $(I, F) \in C_2 \times C_0$ then $I \triangle F \in C_2$
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Clear that $\mathcal{L}(\Gamma) \leq m$. 
Choice of Canonical Paths

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- $\Gamma = \{\gamma_{I,F}\}$

Clear that $\mathcal{L}(\Gamma) \leq m$. One can show that $\varphi(\Gamma) \leq \Delta n^4$. 
Discussion

- The Ising worm process also provides an fpras for the two-point correlation $\text{cov}(\sigma_u, \sigma_v)$ for bounded $d(u, v)$.
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Study higher dimensional spin models using similar methods?